

n -SUBSPACES IN LINEAR AND UNITARY SPACES

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ABSTRACT. We study a relation between brick n -tuples of subspaces of a finite dimensional linear space, and irreducible n -tuples of subspaces of a finite dimensional Hilbert (unitary) space such that a linear combination, with positive coefficients, of orthogonal projections onto these subspaces equals the identity operator. We prove that brick systems of one-dimensional subspaces and the systems obtained from them by applying the Coxeter functors (in particular, all brick triples and quadruples of subspaces) can be unitarized. For each brick triple and quadruple of subspaces, we describe sets of characters that admit a unitarization.

1. INTRODUCTION

A relationship between representations of groups on linear spaces and their unitary representations on Hilbert spaces is useful for the two kinds of representations.

In this paper, we study a relation between brick n -tuples $L = (V; V_1, \dots, V_n)$ of subspaces V_k of a complex finite dimensional linear space V , see Section 2, and irreducible orthoscalar n -tuples $S = (H; H_1, \dots, H_n)$ of subspaces H_k of a finite dimensional Hilbert (unitary) space H , that is, such that there exists a collection of positive numbers $(a_0; a_1, \dots, a_n)$, called a character, such that

$$(1) \quad \sum a_k P_{H_k} = a_0 I,$$

where P_{H_k} are orthogonal projections onto the subspaces H_k and I is the identity operator on H , see Section 4. Recall that an n -tuple $L = (V; V_1, \dots, V_n)$ of subspaces V_k is called brick if any linear operator $X : V \rightarrow V$ such that $X(V_k) \subset V_k$ is a multiple of the identity operator. An n -tuple of orthogonal projections $\{P_{H_k}\}_{k=1}^n$ is called irreducible if, for any linear operator $X : H \rightarrow H$, $[X, P_{H_k}] = 0$, $k = 1, \dots, n$, implies that $X = \lambda I$, $\lambda \in \mathbb{C}$.

If an n -tuple of orthogonal projections P_{H_k} on H is irreducible and satisfies relation (1), then the corresponding collection of the subspaces H_k of the linear space H will be brick, see [KNR]. In this paper, we call a brick collection $L = (V; V_1, \dots, V_n)$ unitarizable if there exists a scalar product on V and a character $\chi = (a_0; a_1, \dots, a_n)$ such that the corresponding collection of orthogonal projections onto $H_k = V_k$ satisfies (1). In Section 4 we prove (Theorems 1 and 2) that brick systems of one-dimensional subspaces and the ones they yield by applying the Coxeter functors (in particular, all brick triples and quadruples of spaces) can be unitarized, see [MS1], [MS2] for unitarizing all nondegenerate brick quadruples with the characters $(\gamma; 1, 1, 1, 1)$, $\gamma > 0$. There are also other examples of systems of subspaces that can be unitarized. In Section 5, for all brick quadruples of subspaces we describe sets of characters that allow for a unitarization, see Theorems 3 and 4.

The interests of the authors to the topics discussed in the paper was increased in connection with the article [EW], where it was remarked that “There seems to be interesting relations with the n -tuples of subspaces and the sums of projections”.

2. ON n -TUPLES OF SUBSPACES OF A LINEAR SPACE

2.1. In this subsection, we recall known facts about n -tuples of subspaces of a linear space, needed in the sequel.

Let $L = (V; V_1, V_2, \dots, V_n)$ be a system of subspaces of V , $\tilde{L} = (\tilde{V}; \tilde{V}_1, \tilde{V}_2, \dots, \tilde{V}_n)$ a system of subspaces of \tilde{V} . A linear operator $R : V \rightarrow \tilde{V}$ is called a homomorphism of the system L into \tilde{L}

if $R(V_i) \subset \tilde{V}_i, \forall i = \overline{1, n}$. $R : V \rightarrow \tilde{V}$ is called an isomorphism if there exists an inverse R^{-1} such that $R^{-1}(\tilde{V}_i) \subset V_i, \forall i = \overline{1, n}$, and the systems L and \tilde{L} will be called isomorphic (equivalent).

Denote by $\text{Hom}(L, \tilde{L})$ the set of homomorphisms from L into \tilde{L} . $\text{End}(L) := \text{Hom}(L, L)$, that is, $\text{End}(L) = \{R : V \rightarrow V \mid R(V_i) \subset V_i, \forall i = \overline{1, n}\}$. A system S is called brick (Schur, transitive) if $\text{End}(L) = \mathbb{C}I$.

Denote by $\text{Idem}(L) = \{R : V \rightarrow V \mid R(V_i) \subset V_i, \forall i = \overline{1, n}, R^2 = R\}$. A system L is called indecomposable if $\text{Idem}(L) = \{0, I\}$. The property of being indecomposable is equivalent to that the system is not isomorphic to a direct sum of two nonzero systems.

It directly follows from the definitions that a brick system is also indecomposable. However, if $n \geq 4$ there are examples showing that the converse is not true.

An isomorphism preserves the property of a system to be indecomposable or brick.

2.2. There are only four indecomposable nonequivalent pairs of subspaces, — $(\mathbb{C}; 0, 0)$, $(\mathbb{C}; \mathbb{C}, 0)$, $(\mathbb{C}; 0, \mathbb{C})$, $(\mathbb{C}; \mathbb{C}, \mathbb{C})$. All of them are brick.

The number of nonequivalent indecomposable triples of subspaces is 9. There are eight triples of subspaces of a one-dimensional space, — $(\mathbb{C}; 0, 0, 0)$, $(\mathbb{C}; \mathbb{C}, 0, 0)$, $(\mathbb{C}; 0, \mathbb{C}, 0)$, $(\mathbb{C}; \mathbb{C}, \mathbb{C}, 0)$, $(\mathbb{C}; 0, 0, \mathbb{C})$, $(\mathbb{C}; \mathbb{C}, 0, \mathbb{C})$, $(\mathbb{C}; 0, \mathbb{C}, \mathbb{C})$, $(\mathbb{C}; \mathbb{C}, \mathbb{C}, \mathbb{C})$, and one triple in a two-dimensional space. This is $(\mathbb{C}^2; \mathbb{C}(1, 0), \mathbb{C}(0, 1), \mathbb{C}(1, 1))$. All of them are brick.

For $n = 4$ already, not every indecomposable n -tuple will be brick. A description of brick quadruples and indecomposable quadruples is given in [B, N, GP] and others. For our purposes, a complete description is not needed, but we will only use some properties.

Let $d = (d_0; d_1, d_2, d_3, d_4)$ be a generalized dimension of the system $L = (V; V_1, V_2, V_3, V_4)$. A Tits form is the quadratic form

$$T(d) = \sum_{i=0}^4 d_i^2 - d_0 \sum_{i=1}^4 d_i.$$

For an indecomposable system L , the Tits form of the dimension d equals either 1 (the dimension d in such a case is called a real root) or 0 (and d is called an imaginary root). If d is a real root, then for this dimension there exists exactly one indecomposable quadruple of subspaces, which is a brick quadruple. If d is an imaginary root, then for this dimension there exists a family of quadruples. Imaginary roots will be multiples of the imaginary root $\sigma = (2; 1, 1, 1, 1)$. In such a case, brick systems will be obtained only for the minimal root σ .

To classify quadruples of spaces, it is convenient to use the notion of a deficiency defined by

$$\text{def}(L) = 2d_0 - \sum_{i=1}^4 d_i.$$

One way to construct indecomposable quadruples is to use Coxeter functors Φ^+ and Φ^- [GP]. These functors allow to use a system to obtain other systems preserving the indecomposability and brick properties. They also preserve the deficiency and the type of the root.

The following is a list of all dimensions corresponding to real roots:

$$\begin{aligned} D_4(2m+1, -1) &= (2m+1; m, m, m, m+1), \text{ def} = -1, \\ D_4(2m+1, 1) &= (2m+1; m+1, m+1, m+1, m), \text{ def} = 1, \\ D_4(2m, -1) &= (2m; m, m, m, m-1), \text{ def} = -1, \\ D_4(2m, 1) &= (2m; m, m, m, m+1), \text{ def} = 1, \end{aligned}$$

and permutation of the subspaces $D_i(\cdot, \cdot), i = 1, 2, 3$;

$$\begin{aligned} D_0(2m+1, -2) &= (2m+1; m, m, m, m), \text{ def} = -2, \\ D_0(2m, 2) &= (2m+1; m+1, m+1, m+1, m+1), \text{ def} = 2, \\ D_{3,4}(2m+1, 0) &= (2m+1; m, m, m+1, m+1), \text{ def} = 0, \end{aligned}$$

and permutations of the subspaces $D_{i,j}(2m+1, 0)$.

In the case where $\text{def} \neq 0$, all indecomposable systems with these dimensions will be brick and can be obtained by applying the Coxeter functors to the simplest collections of subspaces; these are collections of subspaces of a space of dimension 1. For the dimension $D_{i,j}(2m+1, 0)$, a system will be brick only if $m = 0$.

Brick nonequivalent quadruples in a space of dimension $\sigma = (2; 1, 1, 1, 1)$ can be written as follows:

$$\begin{aligned} S_\mu &= (\mathbb{C}^2 = \langle e_1, e_2 \rangle; \langle e_1 \rangle, \langle e_2 \rangle, \langle e_1 + \mu e_2 \rangle, \langle e_1 + e_2 \rangle), \quad \mu \in \mathbb{C} \setminus \{0, 1\}, \\ S_{3,4} &= (\mathbb{C}^2 = \langle e_1, e_2 \rangle; \langle e_1 \rangle, \langle e_2 \rangle, \langle e_1 + e_2 \rangle, \langle e_1 + e_2 \rangle) \end{aligned}$$

and permutations of $S_{i,j}$.

2.3. For $n \geq 5$ a description of indecomposable n -tuples of subspaces is a very difficult problem and contains, as a subproblem, the problem of a description, up to unitary equivalence, of indecomposable pairs of operators on a finite dimensional linear space.

Indeed, let E be a linear space, A, B linear operators on E . Consider quinary of subspaces, $L_{(A,B)} = (E \oplus E; (x, 0), (0, x), (x, x), (x, Ax), (x, Bx), x \in E)$. Such a quinary will be called an operator quinary. Note that each quinary $(V; V_1, \dots, V_5)$ such that $\dim V = 2n$, $\dim V_i = n$, $i = \overline{1, 5}$, and $V_i \cap V_j = 0$, $i \neq j$, is equivalent to an operator quinary with nondegenerate A and B .

A description of indecomposable quinary up to equivalence is already a very difficult problem, — it is the problem of a description of indecomposable pairs of operators on a linear space up to equivalence.

Proposition 2.1. 1) $L_{A,B} \simeq L_{\tilde{A},\tilde{B}} \iff (A, B) \sim (\tilde{A}, \tilde{B})$, that is, there exists an invertible operator T from \tilde{E} into E such that $\tilde{A} = T^{-1}AT$, $\tilde{B} = T^{-1}BT$.
2) $L_{A,B}$ is indecomposable \iff the pair (A, B) is indecomposable, that is, for all idempotents $T = T^2$ on E such that $TA = AT$, $TB = BT$, we have that $T = 0$ or $T = I$.

For $n \geq 5$, the problem of a description of brick n -tuples of subspaces up to equivalence is also very difficult.

Proposition 2.2. $L_{A,B}$ is brick if and only if (A, B) is brick, that is, if $TA = AT$, $TB = BT$, then $T = \lambda I$.

This problem, for example, contains the problem of describing irreducible pairs of unitary operators on a finite dimensional Hilbert space up to unitary equivalence ([MS2]).

2.4. A possible additional condition for the problem of describing indecomposable n -tuples of subspaces to become meaningful is the condition that the subspaces of the collection make a representation of a finite partially ordered set, see papers on representations of partially ordered sets in the category of linear spaces [NR, Kl] and others.

Another additional condition for the problem of a description of irreducible n -tuples of subspaces to become solvable is a condition on possible indecomposable terms in the decomposition of the quadruples $(V; V_{i_1}, V_{i_2}, V_{i_3}, V_{i_4})$, $i_1, i_2, i_3, i_4 \in \{1, 2, \dots, n\}$, $i_k \neq i_j$ ($k \neq j$).

3. ON n -TUPLES OF SUBSPACES OF A HILBERT SPACE

3.1. There are many works dealing with n -tuples $S = (H; H_1, H_2, \dots, H_n)$ of subspaces of a Hilbert space H , see [H, S, EW] and others. In the sequel, H is usually assumed to be a finite dimensional Hilbert space, i.e., a unitary space. Using the Hilbert space property of H we can assign, to every subspace H_i , a unique orthogonal projection $P_i : H \rightarrow H$ onto this subspace. Collections of subspaces $S = (H; H_1, H_2, \dots, H_n)$ and $\tilde{S} = (\tilde{H}; \tilde{H}_1, \tilde{H}_2, \dots, \tilde{H}_n)$ are called unitary equivalent if there exists a unitary operator $U : H \rightarrow \tilde{H}$ such that $UP_i = \tilde{P}_i U \forall i = \overline{1, n}$.

A collection of orthogonal projections $\{P_i\}_{i=1}^n$ on H is called irreducible if for any $X \in L(H)$ satisfying $XP_i = P_i X$ for all $i = \overline{1, n}$ it follows that $X = \lambda I$ ($\lambda \in \mathbb{C}$).

A collection of subspaces $S = (H; H_1, H_2, \dots, H_n)$ of a Hilbert space can always be connected with the collection of subspaces $L = (V = H; V_1 = H_1, V_2 = H_2, \dots, V_n = H_n)$ in the linear space $V = H$, forgetting the scalar product structure. Here unitary equivalent collections will correspond to isomorphic systems in a linear space (an isomorphism of systems of subspaces of a Hilbert space is understood as an isomorphism of the corresponding systems in linear spaces). The converse, of course, is not true.

For a unitary space, one can also talk about brick collections and indecomposable collections of subspaces meaning brick and indecomposable collections of subspaces of H considered as a linear

space. It is easy to see here that indecomposability of a collection of subspaces implies that the corresponding collection of the orthogonal projections is irreducible. If a collection of orthogonal projections is irreducible, then the collection of subspaces need not be, in general, indecomposable. For example, a pair of projections onto two nonorthogonal one-dimensional subspaces in \mathbb{C}^2 will be irreducible but the pair of the corresponding subspaces is decomposable.

3.2. Irreducible pairs of subspaces exist only in one- and two-dimensional unitary spaces. A list of the corresponding unitary nonequivalent pairs of orthogonal projections $\{P_1, P_2\}$ is the following.

- a) $\dim H = 1$: $\{P_1 = 0, P_2 = 0\}$, $\{P_1 = 1, P_2 = 0\}$, $\{P_1 = 0, P_2 = 1\}$, $\{P_1 = 1, P_2 = 1\}$;
- b) $\dim H = 2$:

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} \cos^2 \phi & \cos \phi \sin \phi \\ \cos \phi \sin \phi & \sin^2 \phi \end{pmatrix}, \quad \phi \in (0, \pi/2).$$

A description of triples of subspaces of H up to unitary equivalence is a *-wild problem. Even assuming that two of these spaces are orthogonal, the problem is still *-wild [KS1, KS2].

3.3. For a system of subspaces of a Hilbert space, one can also define Coxeter functors, $\overset{\circ}{F}$, $\overset{\bullet}{F}$, $F^+ = \overset{\circ}{F}\overset{\bullet}{F}$, and $F^- = \overset{\bullet}{F}\overset{\circ}{F}$, see [KRS, Kr], which correspond to the Coxeter functors Φ^+ and Φ^- for systems in a linear space; if a system S in a Hilbert space H corresponds to a system L in a linear space $V = H$, then F^+S is isomorphic to the system Φ^+L , and F^-S is isomorphic to the system Φ^-L .

The Coxeter functors have the following properties: $\overset{\circ}{F} \circ \overset{\bullet}{F} = \overset{\bullet}{F} \circ \overset{\circ}{F} = F^+ \circ F^- = F^- \circ F^+ = \text{Id}$; they preserve the property of the system to be brick or indecomposable, as well as irreducibility. If $d = (d_0; d_1, \dots, d_n)$ is a dimension of a system S , then the dimension of $\overset{\circ}{F}S$ is $\overset{\circ}{c}d := (\sum_{i=1}^n d_i - d_0; d_1, \dots, d_n)$, and the dimension of $\overset{\bullet}{F}S$ is $\overset{\bullet}{c}d := (d_0; d_0 - d_1, \dots, d_0 - d_n)$. Similarly, F^+S has the dimension $c^+d := (\overset{\circ}{c}\overset{\bullet}{c})d \{=\overset{\bullet}{c}(\overset{\circ}{c}(d))\}$, and the dimension of F^-S is $c^-d := (\overset{\circ}{c}\overset{\bullet}{c})d$.

3.4. One of a natural additional condition on the collection $S = (H; H_1, H_2, \dots, H_n)$ so that the problem of unitary description of the n -tuple of subspaces of H becomes solvable is the orthoscalar condition, which is the linear relation $\sum \alpha_k P_{H_k} = \alpha_0 I$ for a fixed character $\chi = (\alpha_0; \alpha_1, \dots, \alpha_n)$, $\alpha_k > 0$ [KRS, Os, OS2, KNR]. A remarkable property of such collections of subspaces is that the Coxeter functors preserve the orthoscalar property, although changing the character, in general. Namely, if S is an orthoscalar collection with a character χ , then $\overset{\circ}{F}S$ is orthoscalar with the character $\overset{\circ}{c}(\chi)$, and $\overset{\bullet}{F}S$ is orthoscalar with the character $\overset{\bullet}{c}(\chi)$.

For $n = 2$ and $n = 3$, orthoscalar collections have a finite Hilbert type, that is, for any fixed character there exists only a finite number of unitary nonequivalent irreducible collections of subspaces satisfying the relation $\sum \alpha_k P_{H_k} = \alpha_0 I$.

For $n = 4$ there are two possibilities that depend on the character. One of them is that there exists a finite number of irreducible unitary nonequivalent such quadruples (all of them are in a finite dimensional H , although their dimensions could increase when changing the characters), and they can be obtained from the simplest ones by applying the Hilbert space version of the Coxeter functors; formulas for orthogonal projections onto the subspaces of such quadruples for $\chi = (\gamma; 1, 1, 1, 1)$, $\gamma > 0$, can be found in [OS1]. Another possibility is a quadruple of one-dimensional subspaces of a two-dimensional space; formulas for the orthogonal projections are given in [KNR].

3.5. For $n \geq 5$, a description, up to unitary equivalence, of n -tuples of subspaces,

$$(H; H_1, H_2, \dots, H_n),$$

such that $\sum \alpha_k P_{H_k} = 2I$ is a *-wild problem [OS1]. It contains the problem of describing triples of orthogonal projections P, Q, R such that $Q \perp R$,

$$P + (I - P) + Q + R + (I - Q - R) = 2I.$$

We remark that for all γ such that $\gamma \in [\frac{n-\sqrt{n^2-4n}}{n}, \frac{n-\sqrt{n^2+4n}}{n}]$, the problem of a unitary description of n -tuples of subspaces satisfying the condition $\sum \alpha_k P_{H_k} = \gamma I$ is not a type I problem, see [KRS, Sh].

3.6. One can impose an additional condition on the orthoscalar collection

$$S = (H; H_1, H_2, \dots, H_n), \quad n \geq 5,$$

which would allow for a description of unitary nonequivalent irreducible collections of subspaces. This is the condition that the subspaces of the collection make a representation of a finite partially ordered set Γ with the number of vertices $|\Gamma| = n$.

In the case where Γ is a primitive partially ordered set, that is, a partially ordered set consisting of k not connected linearly ordered sets $p_1^{(j)} < p_2^{(j)} < \dots < p_{m_i}^{(j)}$, $\sum_{i=1}^k m_i = n$, $p_i^{(j)} \in \Gamma$, $j = 1, \dots, k$, a study of their representations in a Hilbert space is the same as studying collections of subspaces $(H; \{H_i^{(j)}\}_{i=1, \dots, m_j}^{j=1, \dots, k})$ such that $H_i^{(j)} \subset H_{i+1}^{(j)}$ and

$$\sum_{j=1}^k \sum_{i=1}^{m_j} a_i^{(j)} P_{H_i^{(j)}} = I$$

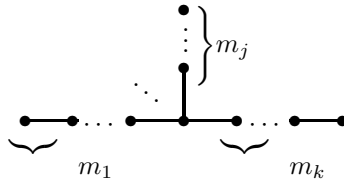
for some fixed collection of positive numbers $\{a_i^{(j)}\}_{i=1, \dots, m_j}^{j=1, \dots, k}$.

Let us now consider the subspaces $\{U_1^{(j)} = H_1^{(j)}, U_2^{(j)} = H_2^{(j)} \ominus H_1^{(j)}, \dots, U_{m_j}^{(j)} = H_{m_j}^{(j)} \ominus H_{m_j-1}^{(j)}, U_0^{(j)} = H \ominus H_{m_j}^{(j)}\}$, $j = \overline{1, k}$, that are mutually orthogonal for fixed j . We have that

$$\sum_{i=0}^{m_j} P_{U_i^{(j)}} = I, \quad j = \overline{1, k}, \quad \sum_{j=1}^k \sum_{i=1}^{m_j} \beta_i^{(j)} P_{U_i^{(j)}} = I,$$

where $\beta_i^{(j)} = a_i^{(j)} + \dots + a_{m_i}^{(j)}$, $i = 1, \dots, m_j$. It is clear that this is the same as to consider a collection of self-adjoint operators $A_j = \sum_{i=1}^{m_j} \beta_i^{(j)} P_{U_i^{(j)}}$, that is, such that the spectrum satisfies $\sigma(A_j) \subset \{0 < \beta_i^{(m_j)} < \dots < \beta_i^{(1)}\}$ and $\sum_{j=1}^k A_j = I$. For a study of such operators, see [KR, VMS, OS2, AOS] and others.

The representation type of such a problem depends on the tree



If the tree corresponds to a Dynkin diagram, then there exists only a finite number of unitary nonequivalent systems of operators $\{A_j\}_{j=1}^k$ (all of them are finite dimensional) for any fixed collection of spectrums; if it is a Euclidean graph, which is an extended Dynkin graph, then depending on the collection of the spectrums, that are the numbers $\{\beta_i^{(j)}\}_{i=1, \dots, m_j}^{j=1, \dots, k}$, the number of such unitary nonequivalent irreducible collections is finite or infinite (all of them are operators on a finite dimensional space, but the dimension of H could increase when changing the admissible spectrums). If this tree contains a Euclidean graph as a proper subgraph, then there always exists a collection $\{\beta_i^{(j)}\}_{i=1, \dots, m_j}^{j=1, \dots, k}$ for which there is an irreducible collection of infinite dimensional operators $\{A_j\}_{j=1}^k$.

3.7. Another additional conditions on the collection $S = (H; H_1, H_2, \dots, H_n)$ for the problem of unitary description to become solvable is to choose a configuration of subspaces with given possible collections $M_{ij} \subset \{(0, 0), (1, 0), (0, 1), (1, 1), 0 < \varphi_1^{(ij)} < \varphi_2^{(ij)} < \dots < \varphi_{m_{ij}}^{(ij)} < \frac{\pi}{2}\}$ of irreducible representations for pairs of subspaces H_i and H_j , $i \neq j$, $i, j = 1, \dots, n$. This is a way one obtains various generalizations of the Temperley-Lieb algebras [G, PSS] and many others.

4. UNITARIZATION

4.1. Definition. We will say that a collection of subspaces $L = (V; V_1, V_2, \dots, V_n)$ of a linear space V can be unitarized with a character $\chi = (a_0; a_1, \dots, a_n)$, $a_0 \geq 0$, $a_k > 0$, $1 \leq k \leq n$, if $H = V$ can be endowed with a scalar product in such a way that $\sum_{k=1}^n a_k P_{H_k} = a_0 I$, where P_{H_k} are the orthogonal projections onto $H_k = V_k$. In other words, the collection L is isomorphic to an orthoscalar collection S with the character χ .

It is clear that a unitarization with a character $\chi = (a_0; a_1, \dots, a_n)$ is the same as a unitarization with the character $\chi' = (a_0/\gamma; a_1/\gamma, \dots, a_n/\gamma)$, $\gamma > 0$.

If $a_0 = 1$, we will write the character $\chi = (a_0; a_1, \dots, a_n)$ as $\chi = (a_1, \dots, a_n)$. Thus, $(a_0; a_1, \dots, a_n) = (\frac{a_1}{a_0}, \dots, \frac{a_n}{a_0})$.

It is clear [KNR] that if an irreducible collection of subspaces of a Hilbert space is orthoscalar, then it is brick. Hence, an indecomposable collection in a linear space can be unitarized if it is brick.

Similarly to [KNR], one can conclude that if for a fixed character χ there exists a unitarization of an indecomposable collection L , then it is unique, that is, if systems of subspaces S and \tilde{S} are orthoscalar with the character χ and are isomorphic to L , then S and \tilde{S} are unitary equivalent.

In this paper we study the following problems.

- 1) For what brick collections there is a unitarization with some character ?
- 2) How to describe the characters that allow for a unitarization of a given brick collection ?

Let us remark that statements connected with the unitarization problem are also contained in [Ki, CBG] and others.

4.2. It is not difficult to get an answer to the above questions for collections of n subspaces if $n = 2$ or $n = 3$.

For $n = 2$, we have the following.

$(\mathbb{C}; 0, 0)$ can be unitarized with the characters $(0; a_1, a_2)$, where $a_1 > 0, a_2 > 0$ are arbitrary positive numbers.

$(\mathbb{C}; \mathbb{C}, 0)$ can be unitarized with the characters $(a_0; a_1, a_2)$, where $a_0 = a_1, a_1 > 0, a_2 > 0$. For $(\mathbb{C}; 0, \mathbb{C})$, the answers are obtained by a corresponding permutation.

$(\mathbb{C}; \mathbb{C}, \mathbb{C})$ can be unitarized with the characters $(a_0; a_1, a_2)$, where $a_0 > 0, a_1 > 0, a_2 > 0, a_0 = a_1 + a_2$.

Let $n = 3$. Note that if one of the subspaces of the collection is zero, then the problem of describing the characters is reduced to the problem with fewer subspaces, since the coefficients corresponding to the zero subspace can be chosen arbitrarily and it does not influence the others. For example, for $(\mathbb{C}; \mathbb{C}, \mathbb{C}, 0)$, we get that a unitarization is only possible with the characters $(a_0; a_1, a_2, a_3)$, where $a_i > 0, 0 \leq i \leq 3, a_0 = a_1 + a_2$.

For $n = 3$ there are only two collections without zero subspaces.

$(\mathbb{C}; \mathbb{C}, \mathbb{C}, \mathbb{C})$ can be unitarized with the characters $(a_0; a_1, a_2, a_3)$, where $a_i > 0, 0 \leq i \leq 3, a_0 = a_1 + a_2 + a_3$.

$(\mathbb{C}^2; \mathbb{C}(1, 0), \mathbb{C}(0, 1), \mathbb{C}(1, 1))$ can be unitarized with the characters $(a_0; a_1, a_2, a_3)$, where $0 < a_i < a_0, 0 \leq i \leq 3, 2a_0 = a_1 + a_2 + a_3$.

4.3. In the general situation, the following propositions are useful when studying unitarization of n -tuples of subspaces.

Proposition 4.1. *A collection of subspaces $L = (V; V_1, \dots, V_n)$ can be unitarized with a character χ if and only if $\Phi^+ L$ and $\Phi^- L$ can be unitarized with the characters $c^- \chi$ and $c^+ \chi$, correspondingly, with the condition that $\Phi^+ L \neq 0$ and $\Phi^- L \neq 0$, correspondingly.*

Proof. Indeed, let L be isomorphic to an orthoscalar collection S with a character χ , and let $\Phi^+ L \neq 0$. Then $F^+ S \neq 0$ and $F^+ S$ is orthoscalar with the character $c^- \chi$. Since $F^+ S$ is isomorphic to $\Phi^+ L$, we see that $\Phi^+ L$ can be unitarized with the character $c^- \chi$. Similarly we obtain that $\Phi^- L$ can be unitarized with the character $c^+ \chi$ if $\Phi^- L \neq 0$. The converse statement follows, since $F^+ F^- = F^- F^+ = \text{Id}$. \square

Proposition 4.2. *If a collection of subspaces, $L = (V; V_1, \dots, V_n)$, of a linear space V can be unitarized with some character, then the collections $L'_0 = (V; V_1, \dots, V_n, 0)$, $L'_1 = (V; V_1, \dots, V_n, V)$, and $L' = (V; V_1, \dots, V_n, V_k)$, $1 \leq k \leq n$, can be unitarized with some characters.*

Proof. Let L be isomorphic to an orthoscalar collection $S = (H; H_1, \dots, H_n)$ with a character $\chi = (a_1, \dots, a_n)$. Then the collection $S'_0 = (H; H_1, \dots, H_n, 0)$ is isomorphic to L'_0 and is orthoscalar with the character $(a_1, \dots, a_n, 1)$, the collection $S'_1 = (H; H_1, \dots, H_n, H)$ is isomorphic to L'_1 and is orthoscalar with the character $(a_1/2, \dots, a_n/2, 1/2)$, the collection $S' = (H; H_1, \dots, H_n, H_k)$ is isomorphic to L' and is orthoscalar with the character $(a_1, \dots, a_k/2, \dots, a_n, a_k/2)$. \square

Theorem 1. *Let H be a linear space of finite dimension m . Let $S = (H; H_1, H_2, \dots, H_n)$ be a brick collection of one-dimensional subspaces of H , that is, $\dim H_i = 1$ (note that brickness is equivalent to indecomposability in this case). Then S is unitarizable with some character.*

Proof. Let us introduce an arbitrary scalar product $(\cdot, \cdot)_1$ on H . Let $T = \sum P_{H_k}$, where P_{H_k} are orthogonal projections onto H_k with respect to the scalar product $(\cdot, \cdot)_1$. Since the collection S is brick, the operator T is nondegenerate. Also, the operator T is nonnegative, being a sum of nonnegative operators. It is clear that T^{-1} is also nonnegative.

Define a new scalar product by $(\cdot, \cdot)_2 = (T^{-1}(\cdot), \cdot)_1$. Such a definition is correct, since T^{-1} is nondegenerate and nonnegative.

Let $v_i \in H_i, i = \overline{1, n}$ ($v_i \neq 0$). Then for all $v \in H$ and all i ,

$$P_{H_i}(v) = \frac{(v, v_i)_1}{(v_i, v_i)_1} \cdot v_i.$$

Let P'_{H_i} be orthogonal projections onto H_i with respect to the scalar product $(\cdot, \cdot)_2$. Then, for all $v \in H$,

$$P'_{H_i}(v) = \frac{(v, v_i)_2}{(v_i, v_i)_2} \cdot v_i = \frac{(T^{-1}(v), v_i)_1}{(v_i, v_i)_2} \cdot v_i = \frac{(v_i, v_i)_1}{(v_i, v_i)_2} \cdot P_{H_i}(T^{-1}(v)).$$

We see that

$$\sum \frac{(v_i, v_i)_2}{(v_i, v_i)_1} \cdot P'_{H_i}(v) = \sum P_{H_i}(T^{-1}(v)) = T(T^{-1}(v)) = v,$$

that is

$$\sum \frac{(v_i, v_i)_2}{(v_i, v_i)_1} \cdot P'_{H_i} = I,$$

which means that the collection S with the character $\chi = \{\frac{(v_i, v_i)_2}{(v_i, v_i)_1}, i = \overline{1, n}\}$ can be unitarized. \square

Theorem 2. 1) *All collections of subspaces of a linear space, which are obtained from brick collections of one-dimensional subspaces by adding its copies (see Proposition 4.2) and by applying the Coxeter functors, can be unitarized with some character.*
2) *In particular, any brick quadruples of spaces can be unitarized with some character.*

Proof. The first part of the theorem follows directly from Propositions 4.1, 4.2, and Theorem 1.

To prove the second part, let us recall (see Section 2) that any brick quadruple has either discrete or continuous spectrum. In the case of a discrete spectrum, the brick quadruples are obtained from the simplest ones by applying the Coxeter functors. Since the simplest quadruples are one-dimensional, they will be unitarizable. Hence, by Proposition 4.1, all brick quadruples, in the case of a discrete spectrum, are also unitarizable. For a continuous spectrum, all brick quadruples have the dimension $(2; 1, 1, 1, 1)$. Hence, they are unitarizable by Theorem 1. \square

5. A DESCRIPTION OF CHARACTERS FOR WHICH REPRESENTATIONS OF A QUADRUPLE OF SUBSPACES OF V CAN BE UNITARIZED

5.1. All brick quadruples of subspaces are of only two types; the generalized dimension is a real root (we will call this case discrete) and an imaginary root (we will call it continuous case). In the discrete case, for every dimension there is exactly one brick quadruple. All possible dimensions in this case can be written as follows:

$$\mathcal{D}_4(2m+1, -1) = (2m+1; m, m, m, m+1), \text{ def} = -1,$$

$$\begin{aligned}
\mathcal{D}_4(2m+1, 1) &= (2m+1; m+1, m+1, m+1, m), \text{ def} = 1, \\
\mathcal{D}_4(2m, -1) &= (2m; m, m, m, m-1), \text{ def} = -1, \\
\mathcal{D}_4(2m, 1) &= (2m; m, m, m, m+1), \text{ def} = 1,
\end{aligned}$$

and permutations of the spaces $\mathcal{D}_i(\cdot, \cdot), i = 1, 2, 3$;

$$\begin{aligned}
\mathcal{D}_0(2m+1, -2) &= (2m+1; m, m, m, m), \text{ def} = -2, \\
\mathcal{D}_0(2m, 2) &= (2m+1; m+1, m+1, m+1, m+1), \text{ def} = 2.
\end{aligned}$$

Theorem 3. *Conditions on the character such that every brick quadruple can be unitarized in the discrete case can be written as follows:*

$$\begin{aligned}
\mathcal{D}_4(2m+1, -1) : & \ m \cdot \text{def}(a) + a_i > 0, \ i = \overline{1, 4}, \ m \cdot \text{def}(a) = a_4 - a_0, \ \text{where } \text{def}(a) = \\
& 2a_0 - a_1 - a_2 - a_3 - a_4, \\
\mathcal{D}_4(2m+1, 1) : & \ a_i - m \cdot \text{def}(a) > 0, \ i = \overline{1, 4}, \ (m+1) \cdot \text{def}(a) + a_4 - a_0 = 0, \\
\mathcal{D}_4(2m, -1) : & \ (m-1) \cdot \text{def}(a) + a_0 - a_i > 0, \ i = \overline{1, 4}, \ m \cdot \text{def}(a) + a_4 = 0, \\
\mathcal{D}_4(2m, 1) : & \ a_0 - a_i - m \cdot \text{def}(a) > 0, \ i = \overline{1, 4}, \ m \cdot \text{def}(a) = a_4, \\
\mathcal{D}_0(4m+1, -2) : & \ m \cdot \text{def}(a) + a_i > 0, \ i = \overline{1, 4}, \ 2m \cdot \text{def}(a) + a_0 = 0, \\
\mathcal{D}_0(4m+1, 2) : & \ a_i - m \cdot \text{def}(a) > 0, \ i = \overline{1, 4}, \ a_0 - (2m+1) \cdot \text{def}(a) = 0, \\
\mathcal{D}_0(4m+3, -2) : & \ m \cdot \text{def}(a) + a_0 - a_i > 0, \ i = \overline{1, 4}, \ (2m+1) \cdot \text{def}(a) + a_0 = 0, \\
\mathcal{D}_0(4m+3, 2) : & \ a_0 - a_i - m \cdot \text{def}(a) > 0, \ i = \overline{1, 4}, \ a_0 - (2m+2) \cdot \text{def}(a) = 0.
\end{aligned}$$

Proof. Using $\overset{\circ}{c}$ and $\overset{\bullet}{c}$ we can write that

$$\begin{aligned}
\mathcal{D}_4(2m+1, -1) &= (\overset{\bullet}{c}\overset{\circ}{c})^{2m} \mathcal{D}_4(1, -1), \\
\mathcal{D}_4(2m+1, 1) &= (\overset{\circ}{c}\overset{\bullet}{c})^{2m} \overset{\bullet}{c} \mathcal{D}_4(1, -1), \\
\mathcal{D}_4(2m, -1) &= (\overset{\circ}{c}\overset{\bullet}{c})^{2m-1} \mathcal{D}_4(1, -1), \\
\mathcal{D}_4(2m, 1) &= (\overset{\circ}{c}\overset{\bullet}{c})^{2m-1} \overset{\bullet}{c} \mathcal{D}_4(1, -1), \\
\mathcal{D}_0(2m+1, -2) &= (\overset{\circ}{c}\overset{\bullet}{c})^m \mathcal{D}_0(1, -2), \\
\mathcal{D}_0(2m+1, 2) &= (\overset{\circ}{c}\overset{\bullet}{c})^m \overset{\bullet}{c} \mathcal{D}_0(1, -2).
\end{aligned}$$

Properties of Coxeter functors show that if there exists a brick collection of subspaces of dimension d , which can be unitarized with a character $\chi = (a_0; a_1, a_2, a_3, a_4)$, then there exists a brick collection of subspaces with the dimension $\overset{\bullet}{c}(d)$, which can be unitarized with the character $\overset{\circ}{c}(\chi)$, as well as a brick collection of subspaces with the dimension $\overset{\circ}{c}(d)$, unitarizable with the character $\overset{\bullet}{c}(\chi)$. Thus, knowing the characters that permit the simplest quadruples to be unitarized, we can find the characters allowing a unitarization of other quadruples in the discrete case.

It is clear that a quadruple with the dimension $\mathcal{D}_4(1, -1) = (1; 0, 0, 0, 1)$ can be unitarized with $\chi_4 = (a_0; a_1, a_2, a_3, a_4)$ if and only if $a_0 = a_4$, $a_i > 0, i = \overline{0, 4}$; a quadruple with the dimension $\mathcal{D}_0(1, -2) = (1; 0, 0, 0, 0)$ can be unitarized with $\chi_1 = (a_0; a_1, a_2, a_3, a_4)$ if and only if $a_0 = 0$, $a_i > 0, i = \overline{1, 4}$.

We get, for example, that for the dimension $\mathcal{D}_4(2m+1, -1) = (\overset{\bullet}{c}\overset{\circ}{c})^{2m} \mathcal{D}_4(1, -1)$, a collection of subspaces can be unitarized with a character χ if and only if $\chi = (\overset{\circ}{c}\overset{\bullet}{c})^{2m} \chi_4$. In other words, if χ is a character that allows for a unitarization of a quadruple with the dimension $(\overset{\circ}{c}\overset{\bullet}{c})^{2m} \mathcal{D}_4(1, -1)$, then $(\overset{\circ}{c}\overset{\bullet}{c})^{2m} \chi$ and χ_4 must satisfy the same conditions.

It is not difficult to calculate that

$$\begin{aligned}
 (\overset{\bullet}{c}\overset{\circ}{c})^{2m}(a_0; a_1, a_2, a_3, a_4) &= (2m \cdot \text{def}(a) + a_0; m \cdot \text{def}(a) + a_1, m \cdot \text{def}(a) + a_2, \\
 &\quad m \cdot \text{def}(a) + a_3, m \cdot \text{def}(a) + a_4), \\
 (\overset{\bullet}{c}\overset{\circ}{c})^{2m} \overset{\bullet}{c}(a_0; a_1, a_2, a_3, a_4) &= (2m \cdot \text{def}(a) + a_0; m \cdot \text{def}(a) + a_0 - a_1, m \cdot \text{def}(a) + a_0 - a_2, \\
 &\quad m \cdot \text{def}(a) + a_0 - a_3, m \cdot \text{def}(a) + a_0 - a_4), \\
 (\overset{\bullet}{c}\overset{\circ}{c})^{2m+1}(a_0; a_1, a_2, a_3, a_4) &= ((2m+1) \cdot \text{def}(a) + a_0; m \cdot \text{def}(a) + a_0 - a_1, \\
 &\quad m \cdot \text{def}(a) + a_0 - a_2, m \cdot \text{def}(a) + a_0 - a_3, m \cdot \text{def}(a) + a_0 - a_4), \\
 (\overset{\bullet}{c}\overset{\circ}{c})^{2m+1} \overset{\bullet}{c}(a_0; a_1, a_2, a_3, a_4) &= ((2m+1) \cdot \text{def}(a) + a_0; (m+1) \cdot \text{def}(a) + a_1, \\
 &\quad (m+1) \cdot \text{def}(a) + a_2, (m+1) \cdot \text{def}(a) + a_3, (m+1) \cdot \text{def}(a) + a_4). \\
 \overset{\circ}{c}(\overset{\bullet}{c}\overset{\circ}{c})^{2m}(a_0; a_1, a_2, a_3, a_4) &= (a_0 - (2m+1) \cdot \text{def}(a); a_1 - m \cdot \text{def}(a), \\
 &\quad a_2 - m \cdot \text{def}(a), a_3 - m \cdot \text{def}(a), a_4 - m \cdot \text{def}(a)), \\
 \overset{\circ}{c}(\overset{\bullet}{c}\overset{\circ}{c})^{2m+1}(a_0; a_1, a_2, a_3, a_4) &= (a_0 - (2m+2) \cdot \text{def}(a); a_0 - a_1 - (m+1) \cdot \text{def}(a), \\
 &\quad a_0 - a_2 - (m+1) \cdot \text{def}(a), a_0 - a_3 - (m+1) \cdot \text{def}(a), \\
 &\quad a_0 - a_4 - (m+1) \cdot \text{def}(a)).
 \end{aligned}$$

By using these formulas and the conditions on the character that allow for a unitarization of the simplest quadruples $\mathcal{D}_4(1, -1)$ and $\mathcal{D}_0(1, -2)$, we obtain conditions on the character for other collections. \square

Let us remark that similar considerations were used in [KPS], although for a different purpose.

Also note that Theorem 3 shows that any brick quadruple with the character $(\gamma; 1, 1, 1, 1)$ can be unitarized in the discrete case, which was proved in [MS2]. Namely, for a quadruple of dimension $d = (d_0; d_1, d_2, d_3, d_4)$, one should take $\gamma = 2 - \text{def}(d)/d_0$. A simple check shows that the conditions of Theorem 3 are satisfied.

5.2. Let us consider the continuous case. Here, brick collections exist only for the dimension $(2; 1, 1, 1, 1)$. There is a series of quadruples parametrized with $\mu \in \mathbb{C} \setminus \{0, 1\}$,

$$S_\mu = (< e_1, e_2 >; < e_1 >, < e_2 >, < e_1 + \mu e_2 >, < e_1 + e_2 >),$$

and two degenerate quadruples,

$$S_{3,4} = (< e_1, e_2 >; < e_1 >, < e_2 >, < e_1 + e_2 >, < e_1 + e_2 >),$$

and $S_{i,j}$ obtained by permutation of the subspaces.

Theorem 4. a) *The degenerate representation of $S_{3,4}$ can be unitarized with the character $\chi = (a_0; a_1, a_2, a_3, a_4)$ if $a_1 + a_2 > a_3 + a_4$, $a_1 < a_2 + a_3 + a_4$, $a_2 < a_1 + a_3 + a_4$, $2a_0 = a_1 + a_2 + a_3 + a_4$, $a_i > 0$.*
 b) *All nondegenerate representations can be unitarized with $\chi = (a_0; a_1, a_2, a_3, a_4)$ if and only if*

$$2a_i < \sum_{j=1}^4 a_j, \quad i = \overline{1, 4}, \quad 2a_0 = \sum_{j=1}^4 a_j, \quad 0 < a_i < a_0.$$

Proof. a) Clearly, the degenerate representation of $S_{3,4}$ can be unitarized with the character $\chi = (a_0; a_1, a_2, a_3, a_4)$ if and only if the representation of the triple of subspaces, $(< e_1, e_2 >; < e_1 >, < e_2 >, < e_1 + e_2 >)$, can be unitarized with the character $(a_0; a_1, a_2, a_3 + a_4)$, that is, if $a_1 + a_2 > a_3 + a_4$, $a_1 < a_2 + a_3 + a_4$, $a_2 < a_1 + a_3 + a_4$, $2a_0 = a_1 + a_2 + a_3 + a_4$, $a_i > 0$.

b) It is clear that the conditions imposed on the character are necessary. Let us prove that they are sufficient.

The proof is similar to considerations in [MOY]. Fix $a_1 \leq a_2 \leq a_3 \leq a_4$, $a_1 + a_2 + a_3 + a_4 = 2$, and use the formulas obtained in [KNR] for solving the equation $a_1 P_1 + a_2 P_2 + a_3 P_3 + a_4 P_4 = I$ in the dimension $(2; 1, 1, 1, 1)$, where P_i are orthogonal projections,

$$\begin{aligned} P_1 &= \frac{1}{2a_1\lambda} \begin{pmatrix} \frac{(\lambda - A)(\lambda + B)}{\sqrt{-(\lambda^2 - A^2)(\lambda^2 - B^2)}} & \sqrt{-(\lambda^2 - A^2)(\lambda^2 - B^2)} \\ -(\lambda + A)(\lambda - B) & \end{pmatrix}, \\ P_2 &= \frac{1}{2a_2\lambda} \begin{pmatrix} \frac{-(\lambda - D)(\lambda + C)}{e^{-ix}\sqrt{-(\lambda^2 - D^2)(\lambda^2 - C^2)}} & e^{ix}\sqrt{-(\lambda^2 - D^2)(\lambda^2 - C^2)} \\ (\lambda + D)(\lambda - C) & \end{pmatrix}, \\ P_3 &= \frac{1}{2a_3\lambda} \begin{pmatrix} \frac{-(\lambda - D)(\lambda - C)}{-e^{-ix}\sqrt{-(\lambda^2 - D^2)(\lambda^2 - C^2)}} & -e^{ix}\sqrt{-(\lambda^2 - D^2)(\lambda^2 - C^2)} \\ (\lambda + D)(\lambda + C) & \end{pmatrix}, \\ P_4 &= \frac{1}{2a_4\lambda} \begin{pmatrix} \frac{(\lambda + A)(\lambda + B)}{-\sqrt{-(\lambda^2 - A^2)(\lambda^2 - B^2)}} & -\sqrt{-(\lambda^2 - A^2)(\lambda^2 - B^2)} \\ -(\lambda - A)(\lambda - B) & \end{pmatrix}, \end{aligned}$$

$A \leq \lambda \leq \min(B, D)$, $0 \leq x \leq 2\pi$, where $A = (a_4 - a_1)/2$, $B = (a_4 + a_1)/2$, $C = (a_3 - a_2)/2$, $D = (a_3 + a_2)/2$.

If $A = 0$, then $a_1 = a_2 = a_3 = a_4 = \frac{1}{2}$. This case was considered in [MS2], where, in particular, it was shown that any brick quadruple can be unitarized in the continuous case with the character $(2; 1, 1, 1, 1)$. So, we assume that $A > 0$.

Let us show that $(\text{Im } P_1, \text{Im } P_2, \text{Im } P_3, \text{Im } P_4)$ give all brick nondegenerate quadruples of the dimension $(2; 1, 1, 1, 1)$ when λ and x are changing.

Denote

$$K_1 = \sqrt{\frac{(\lambda + A)(B - \lambda)}{(\lambda - A)(B + \lambda)}}, \quad K_2 = \sqrt{\frac{(\lambda - C)(D + \lambda)}{(\lambda + C)(D - \lambda)}}, \quad K_3 = \frac{\lambda + C}{\lambda - C} K_2, \quad K_4 = \frac{\lambda - A}{\lambda + A} K_1.$$

Then we have

$$\begin{aligned} \text{Im } P_1 &= \langle e_1 + K_1 e_2 \rangle, & \text{Im } P_2 &= \langle e_1 + e^{-ix} K_2 e_2 \rangle, \\ \text{Im } P_3 &= \langle e_1 - e^{-ix} K_3 e_2 \rangle, & \text{Im } P_4 &= \langle e_1 - K_4 e_2 \rangle. \end{aligned}$$

If $P_i \neq P_j$, $i \neq j$, this system will be isomorphic to the system

$$\langle e_1, e_2 \rangle; \langle e_1 \rangle, \langle e_2 \rangle, \langle e_1 + \mu e_2 \rangle, \langle e_1 + e_2 \rangle,$$

where

$$\mu = \frac{1}{(K_1 + K_4)(K_2 + K_3)} (K_1 K_2 + K_3 K_4 + K_1 K_4 e^{ix} + K_2 K_3 e^{-ix}).$$

Substituting, we get

$$\begin{aligned} \mu &= \frac{1}{2} - \frac{AC}{2\lambda^2} + \frac{1}{4\lambda^2} K_1 K_2^{-1} (\lambda - A)(\lambda - C) e^{ix} + \frac{1}{4\lambda^2} K_1^{-1} K_2 (\lambda + A)(\lambda + C) e^{-ix} \\ &= \frac{1}{2} - \frac{AC}{2\lambda^2} + \frac{1}{4\lambda^2} \sqrt{(\lambda^2 - A^2)(\lambda^2 - C^2)} \left(\sqrt{\frac{(B - \lambda)(D - \lambda)}{(B + \lambda)(D + \lambda)}} e^{-ix} + \sqrt{\frac{(B + \lambda)(D + \lambda)}{(B - \lambda)(D - \lambda)}} e^{ix} \right). \end{aligned}$$

For a fixed λ , we have that μ , considered as a function of x , is an ellipse in \mathbb{C} . If λ increases to $\min(B, D)$, then this ellipse extends to infinity. If λ approaches A , then the ellipse contracts to the point $(\frac{1}{2} - \frac{C}{2A})$. Thus μ can be made arbitrary distinct from 0 and 1 when varying λ and x . \square

Remark. Let us remark that the conditions on the character in part b), Theorem 4, do not depend on the parameter μ , that is, all S_μ , $\mu \in \mathbb{C} \setminus \{0, 1\}$, can be unitarized with the same characters.

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